

Technical Note

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Uncertainty Evaluation via Polynomial Chaos Expansion

Mohammad Mahdi Share Pasand ^{1*}, Ali Tavakoligolpaygani ², Sara Ahmadi ³, Majid Nouri Kamari ⁴

1. Electrical engineering Research Group, Research Center of Technology and Engineering, Standard Research Institute, Karaj, Iran

2. Biomedical Engineering Research Group, Research Center of Technology and Engineering, Standard Research Institute, Karaj, Iran

3. Construction and Mineral Research Group, Research Center of Technology and Engineering, Standard Research Institute, Karaj, Iran

4. Vehicle Engineering Research Group, Research Center of Technology and Engineering, Standard Research Institute, Karaj, Iran

Abstract

Uncertainty evaluation is an important task in almost all activities in the field of calibration and standard conformity assessment. Evaluation of uncertainty is a process in which the statistical characteristics (mainly the first and second moments; i.e. the average and the standard deviation) of the result of a measurement are estimated based on those of contributing factors. The result of this process is used to determine how appropriate a measurement is for a specific application. The measurement uncertainty influences the result of binary decisions made on conformity assessment, calibration and many other applications in the fields of jurisdiction, commerce, law enforcement or healthcare. Another important application of uncertainty evaluation, is to evaluate and validate mathematical or computer models simulating real world phenomena. For example, an uncertainty evaluation process can be utilized to quantify the error characteristics of a simulation software. Many techniques have been introduced and practiced for uncertainty evaluation. The applicability of each method relies upon the measurement model, possibility of performing repeated simulations and availability of statistical data. In this paper, we examine a celebrated uncertainty propagation method, namely; the Polynomial Chaos Expansion in the uncertainty evaluation of measurements described by a rigorous mathematical model. These measurement processes have been conventionally studied using sensitivity coefficients which can be calculated via partial derivatives. The two approaches are compared in an illustrative example.

Keywords Uncertainty evaluation, Measurement Uncertainty, Polynomial Chaos Expansion.

1. Introduction

Uncertainty evaluation or quantification is the process of attributing a quantitative description to the uncertainty of a numerical result. Uncertainty may be associated with measurement, numerical calculation, estimation, simulation and any form of numerical declaration [1]. Uncertainties of measurement stem from laboratory personnel errors [2], measurement inaccuracies attributed to devices

or methods in their ideal performance [3], calibration errors [2], innate uncertainties originated from the definition of measurands [4] as well as many other sources. The value of measurement uncertainty can't be known for sure due to its nature. However, the uncertainty can be modelled to some extent by statistical random variables. In this way, a measurement uncertainty may be described as a random variable with a known probability distribution



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* Correspondence:
sharepasand@standard.ac.ir

function or at least known statistical moments (e.g. mean, variance, etc.) [5, 6 and 7]. For instance, a measurement uncertainty may be described as being a random variable with uniform distribution function between $[a,b]$. This means one can expect the value of the uncertainty will lie in the interval $[a,b]$ in which the probability of having any of values in the interval are the same. This also implies that there is zero (or practically negligible) possibility that the aforementioned measurement uncertainty takes a value outside the interval $[a,b]$. The measurement uncertainty is formed by many contributing factors which are usually listed in an uncertainty budget [1]. The contributing factors, are uncertainties in the measurement process which affect the eventual uncertainty attributed to the measurement result. For instance, if the room temperature affects the measurement result for the length of a metallic rod, then the room temperature is a contributing factor in the measurement uncertainty of the rod length. To identify the contributing factors requires laboratory/field experimental knowledge as well as metrological expertise on the measurement principle [1, 6 and 7]. Several examples are given in [1, 6 and 7] regarding the identification of contributing factors and quantification of their effect on the measurement uncertainty. In many practical situations, the probability distribution function of a measurement uncertainty is not known. However, the probability density functions of contributing factors can be determined reliably.

This paper describes the application of a celebrated method known as Polynomial Chaos Expansion (PCE) to the well-established discipline of uncertainty evaluation. The PCE method was originally proposed by Norbert Weiner in 1938 and has been used in many application areas [5, 8 and 9]. PCE has several applications. For example, it may be used to numerically solve complex differential equations described by random parameters. However, the most established usage of PCE is the uncertainty evaluation in partial and ordinary differential equations [9-24]. This paper approaches the problem as a laboratory measurement experiment. It first revisits the well-established Gum method for sensitivity analysis which has been used for uncertainty evaluation in static systems for decades. Then, the PCE will be introduced for both static and dynamic systems. This paper focuses on time invariant dynamic systems described by linear and nonlinear time invariant ordinary differential equations. Partial differential equations lie beyond the scope of the present paper. The novelties of the paper are as follows:

- A formal state space representation is given for linear time invariant dynamic systems with statistical parameter uncertainties via PCE.
- The PCE is utilized to quantify uncertainty of measure-

ment in a manner similar to that of conventional well-established method of GUM. The two approaches are compared in an illustrative example.

The paper is organized in the following sections: In the first section the Gum method of [4] is introduced. In the second section, the PCE method is introduced. The application of PCE to systems described by nonlinear and linear time invariant ordinary differential equations is discussed in the next sections. An example is included to illustrate the provided subjects. The two last sections are dedicated to results and discussion as well as conclusion.

2. Preliminaries

In this section, background material and preliminaries of the paper are provided. It includes the introduction of the GUM and PCE methods.

2.1. The GUM method

In the ISO Guide 98-1 [4] known as the Guide on Uncertainty in Measurement (GUM), a formal method is prescribed for uncertainty evaluation. To illustrate this method, consider the following measurement model:

$$y = f(x_1, x_2, \dots, x_N) \quad (1)$$

In which y is the measurand and $x_1, x_2, \dots, x_N$ are the N quantities contributing or affecting the measurement result and the term $f(\cdot)$ represents a known multi-input function. In the relevant context, it is assumed that the function $f(\cdot)$ admits a Taylor series expansion. For example, in the measurement of a person's weight (y), resolution and accuracy of the weighing apparatus at nominal environmental temperature may be considered as contributing factors (x_1, x_2). Another factor (x_3) is the environment temperature deviation from nominal value (e.g. 23 degrees Celsius) which affect the accuracy of the weighing process. The list may seem endless, however those factors having the most contributions on the measurement uncertainty are usually finite and are known to the experts of the field as illustrated in several examples in [1], [6] and [7].

Equation (1) gives an explicit expression stating how contributing factors make the measurand value. Each contributing factor as well as the measurand are assumed as random variables. Since the function $f(\cdot)$ is assumed to be known exactly, taking expectations from the both sides of (1), one obtains:

$$\mu_y = E\{f(x_1, x_2, \dots, x_N)\} \quad (2)$$

In which $E\{\cdot\}$ represents the expected values of $x_1, x_2, \dots, x_N$ and y respectively. In uncertainty evaluation, the devi-

ation of the measurand from its average is of most concern. This deviation can be calculated using (1) and (2) as follows:

$$y - \mu_y = f(x_1, x_2, \dots, x_N) - \mu_y \quad (3)$$

The deviation of the measurand from its expected value is usually small enough to allow approximation of (3) via the first order terms of the Taylor series expansion of the function $f(\cdot)$ as follows:

$$y - \mu_y \approx \sum_{i=1}^N \frac{\partial f}{\partial x_i} \bigg|_{\mu_{x_1}, \dots, \mu_{x_N}} (x_i - \mu_{x_i}) \quad (4)$$

In which $\partial f / \partial x_i$ is the partial derivative of the function $f(\cdot)$ that should be evaluated by substituting the expected values of $x_1, x_2, \dots, x_N$ after derivation. Squaring the both sides of (4) results:

$$\begin{aligned} (y - \mu_y)^2 &\approx \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i} \right)^2 (x_i - \mu_{x_i})^2 + \dots \\ &2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} (x_i - \mu_{x_i}) (x_j - \mu_{x_j}) \end{aligned} \quad (5)$$

Taking expectations from the both sides of (5) and using the definition of standard deviation, variance and correlation coefficient leads to the following:

$$\begin{aligned} \sigma_y^2 &\approx \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \dots \\ &2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} r(x_i, x_j) \sigma_{x_i} \sigma_{x_j} \end{aligned} \quad (6)$$

In which $\sigma_{x_1}, \dots, \sigma_{x_N}$ and σ_y represent standard deviations of contributing factors and the measurand, $r(x_i, x_j)$ is the correlation coefficient between x_i and x_j . If all contributing factors can be assumed mutually uncorrelated, (6) is reduced to (7) as follows:

$$\sigma_y^2 \approx \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 \quad (7)$$

Equations (6) and (7) are known as uncertainty propagation equations and are widely used in uncertainty evaluation of measurements [1, 4, 6 and 7]. Note that in [4] and similar references, the variance of measurand is shown by u_y^2 instead of σ_y^2 . However, as a matter of notational simplicity, we will use the latter throughout this paper. It is also important to note that what we know as the

uncertainty is in fact the so-called expanded uncertainty [4] which is the standard deviation of the measurand σ_y (equivalently u_y) multiplied by a coverage factor k [1, 4]. The coverage factor for Gaussian distribution with 95% confidence interval is $k=2$ which results in an expanded uncertainty of double magnitude compared to the standard uncertainty [1].

Equation (1) is restricted to static systems in which the influencing factors $x_1, x_2, \dots, x_N$ as well as the measurand y have static, constant values. It is possible to deploy (1) to describe the steady state behavior of an originally dynamic equation as well. However, this is acceptable only if the transient response of the influencing factors and measurand (e.g. over shoot, settling time, etc.) is not of concern. See [1] and references therein for more information regarding the transient and steady state responses. To address dynamic systems, it is necessary to formulate the measurand as the output of a differential equation. This is not addressed in [3] and can't be dealt with via partial derivatives as in (4)-(5). This motivates the use of more advanced techniques including the so-called Polynomial Chaos Expansion (PCE). The next section introduces this method.

2.2. Polynomial Chaos Expansion

In this section, a simple version of PCE is briefly discussed. It is first applied to the static system (1) to demonstrate its core idea. Then PCE is introduced for dynamic systems described by ordinary differential equations. PCE formulates each contributing factor in (1) as a polynomial consisting of orthogonal terms as follows:

$$x_i = \sum_{j=0}^{N_i} \hat{x}_{ij} \varphi_{ij}(\varepsilon_i) \quad (8)$$

In which $\hat{x}_{ij}$ is a coefficient to be determined in the process of PCE method. Functions $\varphi_{ij}(\cdot)$ are mutually orthogonal polynomial terms and ε_{ij} is a random variable of known probability density function. Except for $\varphi_{i0}(\cdot)$ which is always unity, all of $\varphi_{ij}(\varepsilon_i)$ are zero mean functions of the zero mean random variables ε_i . The integer N_i is the number of terms required for estimating x_i via the summation. In the generalized PCE, it is assumed that all random variables are independent and identically distributed with specific distribution functions. In the PCE procedure, we start with (8) and use the measurement mathematical description to attain the coefficients of a PCE description for the measurand. Orthogonality of polynomials simplifies the derivation of the coefficients using the so-called Galerkin projection [9].

Table 1 [15] gives orthogonal polynomials for frequently used random variables. If these polynomials are chosen

properly in relation to the random variable's probability distribution functions, the number of required terms in (8) will be reduced significantly [9]. This is illustrated in the example provided in the present paper. However, in the most generic setting, an infinite number of terms may be required. In these situations, a truncated series is used instead of the infinite series. The truncation error upper bound should be then considered in the uncertainty evaluation analysis. The first two terms of some of these polynomials are given in Table 2. Higher order terms are derived using recurrence formulas given for instance in [15] and elsewhere.

Table 1. Orthogonal polynomials in PCE [15]

Continuous	Random variable	Orthogonal polynomial
	Gaussian (Normal)	Hermite
	Gamma	Laguerre
	Betta	Jacobi
	Uniform	Legendre
Discrete	Poisson	Charlier
	Binomial	Krawtchouk
	Negative Binomial	Meixner
	Hypergeometric	Hahn

Table 2. Orthogonal polynomials first and second basis functions

Polynomial	$\varphi_{i0}(\varepsilon)$	$\varphi_{i1}(\varepsilon)$	Parameters
Hermite	1	ε	
Laguerre	1	$1 - \varepsilon$	α
Jacobi	1	$\alpha + 1 + 0.5(\alpha + \beta + 2)(\varepsilon - 1)$	α, β
Legendre	1	ε	
Charlier	1	$(\varepsilon - \alpha)/\alpha$	α
Krawtchouk	1	$-2\varepsilon + n$	n
Meixner	1	$\alpha + \varepsilon(1 - 1/\beta)$	α, β
Hahn	1	$\varepsilon(\alpha + \beta + 2) - (n - 1)(\beta + 1)$	α, β, n

In addition, each two terms of these functions are mutually un-correlated. For example, if the random variable has a Uniform distribution function, then $\varphi_{ij}(\cdot)$ can be chosen from a Legendre polynomial basis. According to (8), x_i is a random variable the statistical moments of which can be stated as follows:

$$\mu(x_i) = \hat{x}_{i0} \quad , \quad \sigma_{x_i}^2 = \sum_{j=0}^N \hat{x}_{ij}^2 \quad (9)$$

Substituting y and x_i in (1) by their corresponding polynomial expansions one obtains:

$$y = \sum_{j=0}^N \hat{y}_j \gamma_j(\varepsilon_y) = f\left(\sum_{j=0}^{N_1} \hat{x}_{1j} \varphi_{1j}(\varepsilon_1), \dots, \sum_{j=0}^{N_N} \hat{x}_{Nj} \varphi_{Nj}(\varepsilon_N)\right) \quad (10)$$

Therefore:

$$y - \mu_y = \sum_{j=1}^{H_i} \hat{y}_j \gamma_j(\varepsilon_j) = f\left(\sum_{j=0}^{N_1} \hat{x}_{1j} \varphi_{1j}(\varepsilon_1), \dots, \sum_{j=0}^{N_N} \hat{x}_{Nj} \varphi_{Nj}(\varepsilon_N)\right) - E\left\{f\left(\sum_{j=0}^{N_1} \hat{x}_{1j} \varphi_{1j}(\varepsilon_1), \dots, \sum_{j=0}^{N_N} \hat{x}_{Nj} \varphi_{Nj}(\varepsilon_N)\right)\right\} \quad (11)$$

Following computational steps demonstrated in (4)-(7), one obtains:

$$y - \mu_y \approx \sum_{i=1}^N \frac{\partial f}{\partial x_i} \sum_{j=0}^{N_i} \hat{x}_{ij} \varphi_{ij}(\varepsilon_i) - E\left\{\sum_{i=1}^N \frac{\partial f}{\partial x_i} \sum_{j=0}^{N_i} \hat{x}_{ij} \varphi_{ij}(\varepsilon_i)\right\} \quad (12)$$

Therefore:

$$(y - \mu_y)^2 \approx \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}\right)^2 \left(\sum_{j=0}^{N_i} \hat{x}_{ij} \varphi_{ij}(\varepsilon_i)\right)^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \left(\sum_{k=0}^{N_i} \hat{x}_{ik} \varphi_{ik}(\varepsilon_i)\right) \left(\sum_{k=0}^{N_j} \hat{x}_{jk} \varphi_{jk}(\varepsilon_j)\right) \quad (13)$$

Taking expectations from the both sides and using the definition of standard deviation, variance and correlation coefficient leads to the following:

$$\sigma_y^2 \approx \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} r(x_i, x_j) \sigma_{x_i} \sigma_{x_j} \quad (14)$$

This conforms to (6). In other words, if one uses the first order terms in the Taylor series expansion of $f(\cdot)$, the result will be the same for the GUM and PCE methods.

2.3. PCE for Measurements described by ordinary differential equations

There are physical and social phenomena which can't be described using (1). For instance, if a measurand is the dynamic response of an ordinary differential equation, it may not be possible to obtain an explicit equation like (1) to describe it. In these situations, using the ordinary differential equation might be a better choice. Consider the following dynamic equation describing a measurand:

$$\dot{x} = G(x, v), \quad y = h(x) \quad (15)$$

In which $x = [x_1, \dots, x_N] \in \mathbb{R}^N$ is a vector consisting of all factors x_i . Vector $v \in \mathbb{R}^M$ is the exogenous input. Functions $G(\cdot) \in \mathbb{R}^{N \times N}$ and $h(\cdot) \in \mathbb{R}^{1 \times N}$ are the state and output functions of the ordinary differential equation (15). Equation (15) is different from (1) in two aspects: Firstly, (15) yields a dynamic time varying response for the measurand while (1) yields a stationary value. Secondly, (15) is not an explicit expression of the measurand. In fact, to obtain and explicit statement for the measurand, one has to solve equation (15) analytically. If the functions $G(\cdot)$, $h(\cdot)$ are linear or have specific properties, it is straightforward to derive the solution to (15). However, for other functions it is not easy to obtain the equation output explicitly. In these cases, (15) should be solved numerically. Note that:

$$\mu_x = E\{G(x, v)\} \quad (16)$$

$$\mu_y = E\{h(x)\} \quad (17)$$

$$y - \mu_y = h(x) - E\{h(x)\} \quad (18)$$

To give a general formulation of the PCE problem for the nonlinear system is not straight forward since the derivations depend on the expected values of the nonlinear functions. We therefore address the case of linear time invariant systems in the following section. The inaccuracies arising from linearized model will be assumed as uncertainties.

3. The Proposed method

Equations (15)-(18) are the starting point for the uncertainty propagation analysis in nonlinear dynamical systems. To provide a global result for all types of nonlinear functions is complicated. However, in many situations, a credible linear approximation exists for the measurement system (15). In fact, many measurement systems

are modified or amended by additional signal conditioning circuitry to make the overall system behavior linear and time invariant [1]. Therefore, it is of common interest in many engineering applications to consider linear time invariant systems. If the original system exhibits highly nonlinear behavior in some situations, one may consider the nonlinearity as a contributing factor to the measurement uncertainty [6, 7]. Therefore, the nonlinearity may be modelled on its own right as a random variable with prescribed bounds representing the maximum expected deviation of the nonlinear original system response from that of the linear model. Many systems can be adequately modelled via linear time invariant models. Those systems for which the linear time invariant model is not adequate, the system state variations in the vicinity of an equilibrium point may be modelled via a linear time invariant system as long as the response remains close to the equilibrium [25]. Therefore, we proceed by assuming a linear time invariant system as a special case of (15) as follows:

$$\dot{x} = Ax + Bv, \quad y = Cx \quad (19)$$

Equations (19) can be expressed in terms of (15) using the first term of the Taylor series expansion of the nonlinear vector functions and defining the differences between the states and their equilibrium values. The

$$A = \frac{\partial G}{\partial x} \big|_{x=x^*}, \quad B = \frac{\partial G}{\partial v} \big|_{x=x^*}, \quad C = \frac{\partial h}{\partial x} \big|_{x^*} \quad (20)$$

To accommodate for the uncertainties arising from linearization as well as other parametric inaccuracies we define the system matrices in (19) via PCE as follows:

$$A = \sum_{j=0}^N \hat{A}_j \varphi_{Pj}(\varepsilon_P) = \hat{A}_a \Phi_P^T$$

$$\Phi_P = [\varphi_{P0} I_{N \times N} \quad \dots \quad \varphi_{PN} I_{N \times N}]$$

$$\hat{A}_a = [\hat{A}_0 \quad \dots \quad \hat{A}_N], \quad \hat{B}_a = \begin{bmatrix} \hat{B}_0 \\ \vdots \\ \hat{B}_N \end{bmatrix}$$

$$B = \sum_{j=0}^N \hat{B}_j \varphi_{Pj}(\varepsilon_P) = \Phi_P \hat{B}_a \quad (21)$$

$$C = \sum_{j=0}^{N_B} \hat{C}_j \varphi_{Pj}(\varepsilon_P) = \hat{C}_a \Phi_P^T$$

$$\hat{C}_a = [\hat{C}_0 \quad \dots \quad \hat{C}_N] \quad (22)$$

In which $\hat{A}_p, \hat{B}_j$ and $\hat{C}_j$ are matrix coefficients for the PCE to be determined, $\varphi_{p_j}(\cdot)$ is the j^{th} basis function of the family of orthogonal polynomials corresponding to the probability density functions of random variable ε_p . For parametric uncertainties uniform and normal distribution functions are common choices whose corresponding polynomials are given in Table 1.

Note that all entries of the matrix Φ_p are polynomial basis terms of random variables. Therefore, this matrix does not vary with time. As in many practical situations, the exogenous input is not exactly known as well. For instance, if the input is measured, it is certainly subject to measurement inaccuracies and noises. Also, the input may model some partially known environmental contributing factors which is uncertain to some extent. Therefore, we model the input in a manner similar to that of the state variable vector. Define:

$$x_i = \sum_{j=0}^N \hat{x}_{ij} \varphi_{pj}(\varepsilon_p) \rightarrow x = \Phi_p \hat{x}_a$$

$$\hat{x}_a = \begin{bmatrix} \hat{x}_{11} \\ \vdots \\ \hat{x}_{N1} \\ \vdots \\ \hat{x}_{N1} \\ \vdots \\ \hat{x}_{NN} \end{bmatrix} \quad (23)$$

In which $\hat{x}_{ij}$ is a coefficient to be determined in the process of PCE method. Taking time derivative of (21) and using (19), (22) one obtains:

$$\dot{x} = \hat{A}_a \Phi_p^T x + \Phi_p \hat{B}_a v \quad (24)$$

Substituting (23) into (24) one has:

$$\Phi_p \dot{\hat{x}}_a = \hat{A}_a \Phi_p^T \Phi_p \hat{x}_a + \Phi_p \hat{B}_a v \quad (25)$$

Note that:

$$\Phi_p^T \Phi_p = \begin{bmatrix} \varphi_{p0}^2 I_{N \times N} & \dots & \varphi_{p0} \varphi_{pN} I_{N \times N} \\ \vdots & \ddots & \vdots \\ \varphi_{p0} \varphi_{pN} I_{N \times N} & \dots & \varphi_{pN}^2 I_{N \times N} \end{bmatrix} \quad (26)$$

Multiplying both sides of (25) into a polynomial basis term $\varphi_{p_j^*}(\varepsilon_p)$ with a specific j^* , taking expectations and using the mutual orthogonality assumption, (25) becomes:

$$\begin{bmatrix} \hat{x}_{1j^*} \\ \vdots \\ \hat{x}_{Nj^*} \end{bmatrix} = \hat{A}_a E\{\varphi_{p_j^*} \Phi_p^T \Phi_p\} \hat{x}_a + \hat{B}_{j^*} v \quad (27)$$

Define:

$$E\{\varphi_{p_j} \Phi_p^T \Phi_p\} = \Psi_j, \quad \Psi = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{bmatrix} \quad (28)$$

Therefore:

$$\dot{\hat{x}}_a = \hat{A}_a \Psi \hat{x}_a + \hat{B}_a v \quad (29)$$

Note that (29) is a linear time invariant state space differential equation with higher dimensions compared to (19).

4. Illustrative Example

Consider the discharge voltage of a capacitor described by the following equation:

$$v(t) = \exp(-pt) \quad (30)$$

In which $p > 0$ is the circuit pole (inverse of the time constant), $\tau > 0$ is the time and $\exp(\cdot)$ represents the exponential function. Assume the system pole is a random variable with uniform distribution, i.e.:

$$p \sim U(0.9, 1.1) \quad (31)$$

The expected value for the voltage (30) as a random variable can be given as:

$$\mu_v(t) = E\{v(t)\} = \int_{-\infty}^{\infty} v(t) f_p(p) dp \quad (32)$$

In which the probability distribution function of p is represented by $f_p(p)$. Substituting the distribution function with that of a uniform random variable and taking the integral (32), one obtains:

$$\mu_v(t) = \frac{5}{t} (\exp(-0.9t) - \exp(-1.1t)) \quad (33)$$

The variance can be obtained in a similar fashion as:

$$\sigma_v^2(t) = E\{v^2(t)\} - \mu_v^2 = \frac{25}{t} (\exp(-0.9t) - \exp(-1.1t)) \times \left(\left(\frac{1}{10} - \frac{1}{t} \right) \exp(-0.9t) + \left(\frac{1}{10} + \frac{1}{t} \right) \exp(-1.1t) \right) \quad (34)$$

Applying the PCE method with two terms of Legendre polynomials (corresponding to Uniform random variables) one has:

$$p = p_0 + p_1 \varepsilon = 1 + \frac{\varepsilon}{10\sqrt{3}}, \quad v = v_0 + v_1 \varepsilon \quad (35)$$

Note that p_0 and p_1 are the mean and standard deviation of p respectively. The uniform random variable is described in (31). Hence the values given in (35). The voltage waveform (30) is the response of the following differential equation:

$$\dot{v}(t) = -pv(t), \quad v(0) = 1^v \quad (36)$$

Substituting (35) into (36), taking expectations and noticing that ε is zero-mean, (36) results the following:

$$\begin{cases} \dot{v}_0(t) = -p_0 v_0(t); v_0(0) = 1 \\ \dot{v}_1(t) = -p_0 v_1(t) - p_1 v_0(t); v_1(0) = 0 \end{cases} \quad (37)$$

The first equation has the following explicit solution:

$$v_0(t) = \exp(-p_0 t) \quad (38)$$

Using (38) in (37) results:

$$\dot{v}_1(t) = -p_0 v_1(t) - p_1 \exp(-p_0 t); v_1(0) = 0 \quad (39)$$

Equation (39) is a linear time invariant ordinary differential equation with exogenous input with the following solution:

$$v_1(t) = -p_1 t \cdot \exp(-p_0 t) \quad (40)$$

Using (39) and (40) in (35), one derives the PCE description for the voltage (3). The expected value and the variance are:

$$\mu_{v,PCE} = v_0(t) \quad (41)$$

$$\sigma_{v,PCE}^2 = v_1^2(t) \quad (42)$$

In which $\mu_{v,PCE}$ and $\sigma_{v,PCE}^2$ are the approximate mean and variance of the voltage computed via the PCE with two terms. The error between the approximated mean and variance and their corresponding actual values can be computed by subtracting (41) from (33) and (42) from (34). The following figures show actual values for the mean and variance and the approximated values derived from PCE.

Using the GUM [4] method, one obtains:

$$\frac{\partial v(t)}{\partial p} \Big|_{p_0} = -t \exp(-p_0 t) \quad (43)$$

Therefore using (14):

$$\mu_{v,GUM} = -t \exp(-p_0 t) \mu_p, \quad (44)$$

$$\sigma_{v,GUM}^2 = t^2 \exp(-2p_0 t) \sigma_p^2 \quad (45)$$

In which $\sigma_{v,GUM}^2$ is the approximate variance derived via GUM method and σ_p^2 is the variance of the random variable p for which one has:

$$\mu_p = p_0 = 1, \quad \sigma_p^2 = p_1^2 = \frac{1}{300} \quad (46)$$

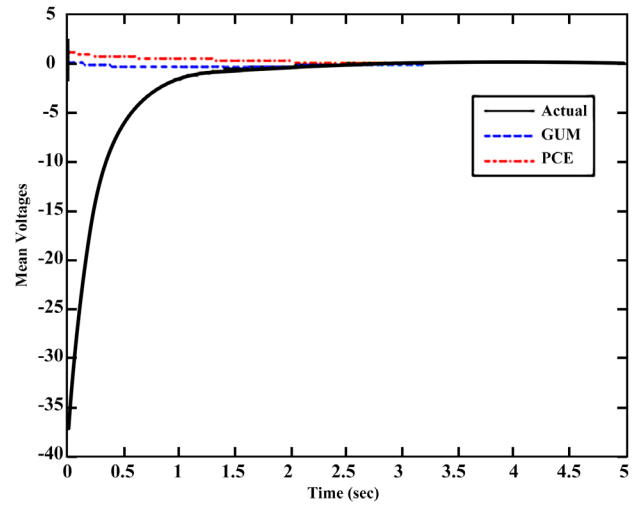


Fig 1. The actual and approximated mean voltages

Figure 1 shows the mean values of voltages. The black curve shows the waveform of the actual mean voltage given by (33). The blue dash curve is the approximate GUM mean given by (44). The red dash-dot curve is the mean approximated via PCE method which is given by (40) and (38). Both methods give inaccurate accounts of the mean value in the transient mode. The both approximations become more accurate when the response reaches its steady state.

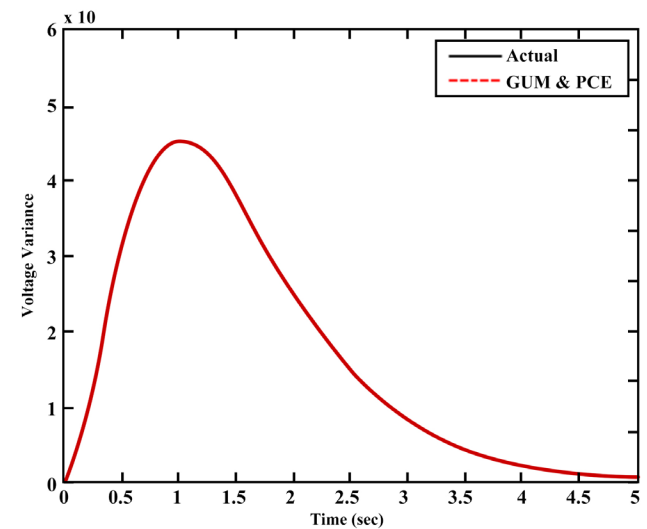


Fig 2. The actual and approximated variance of voltage

Figure 2, shows the actual and approximate variances. The Gum and PCE yields exactly the same expressions for variances (See (45) and (40), (42)). The actual variance is given in (34) and differs from the approximations. However, the error is very small as can be observed from Figure 2.

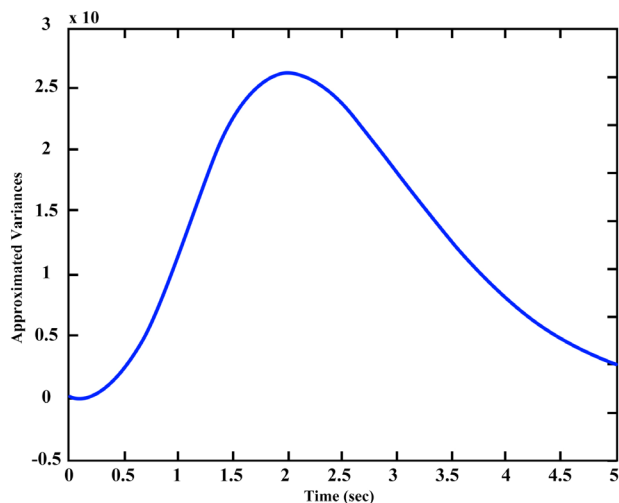


Fig 3. Error between the actual and approximated variances

Figure 3 shows the error between actual variance and the approximated one which is of order less than 1% compared to the actual value. In practical situations like those encountered in uncertainty evaluation of laboratory measurements [1], we are mostly interested in variance approximations. The transient response of statistical measures is usually of less importance while the steady state should be approximated with high accuracy.

This example has the advantage of having the explicit expressions for mean and variances (See (33) and (34)). In most cases, this expression can't be obtained analytically. Also, in this example, the explicit solution to the PCE coefficients could be derived using basic tools from the theory of ordinary differential equations. For more advanced problems including linear time invariant equations with higher dimensions, linear time varying equations and nonlinear equations, both the differential equation and the PCE formulation should be solved numerically. In these situations, the main advantage of PCE in relation to GUM emerges. When the explicit solution to the differential equation is not known analytically, it is not possible to apply the GUM method since this method requires an algebraic formula to compute partial derivatives. However, PCE can still be used without any additional requirement. In fact, PCE can be efficiently used to numerically quantify the uncertainties in nonlinear differential systems of higher dimensions as well as linear

ones. For more simplistic problems, the accuracy of PCE can be improved by utilizing polynomials of more than two terms in (35).

5. Results and Discussion

The linear time invariant dynamics of a dynamic measurement system is transformed into another linear time invariant dynamics after applying the polynomial chaos expansion. The resulting dynamics has increased dimensions. Solving the results dynamics, will yield approximations of the measurand mean, variance and other statistical moments. The main advantages of PCE for uncertainty evaluation are as follows:

Firstly, several contributing factors can be analyzed simultaneously. This is a huge advantage compared to conventional sensitivity analysis in which the effect of a contributing factor can only be evaluated with other factors assumed deterministic.

Secondly the resulting system of equations for PCE is linear time invariant if the original system is linear time invariant.

The main drawback of PCE is its numerical inefficiency for very large dimensions and high orders of polynomials. However, there are several practical situations in which the dimensions of the problem can be reduced significantly. These cases are as follows:

- 1- In some cases, many contributing uncertainties exist with minor effects on the result. Therefore, PCE can be run for each feasible set of uncertainties. Then those contributing factors with minor effect on the result can be eliminated. The resultant problem will have reduced dimensions.
- 2- In many practical cases, the number of terms for the PCE summation can be reduced to finite values as small as 2 (See the example).

Conclusion

In this paper the polynomial chaos expansion method is revisited and its utilization to evaluate the measurement uncertainty in dynamical systems is elaborated. The method is compared to the well-established uncertainty evaluation method of GUM in an example. A formal state space description is given for PCE applied to linear time invariant systems as well.

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Declarations

Ethics approval and consent to participate

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Consent for publication

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Competing interests

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